Derivation of the analytical solution for torque free motion of a rigid body

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Abstract

In this report the rotational motion of a single rigid body in the absence of external forces and torques is considered. The dynamic equations governing this motion are integrated in closed form. Only the general case of an unsymmetrical rigid body is considered. A Matlab implementation is available at [1].

I. Note

We could find two textbooks [2], [3] that contain a detailed derivation of the analytical solution for torque free motion of a rigid body (the one in [3] will be adopted here). The solution involves Jacobi elliptic functions, and a brief introduction is included in Appendix 1.

II. List of variables used

All the variables used in this report are expressed in a body fixed coordinate frame. This frame is chosen in such a way that just three scalars (principal moments of inertia) are sufficient to describe the inertial characteristics of the body ([5] pp. 488).

J_1, J_2 and J_3 are the principle moments of inertia. It is assumed that, $J_1 > J_2 > J_3$ [1]. ω_1, ω_2 and ω_3 are angular velocities and a "hat" above them will indicate an initial condition ($\hat{\omega}_1$), T is rotational kinetic energy, and h is the length of the angular momentum vector. Note that, T and h are constant due to the assumption of zero external torques. In the derivation below we will make use of the ratio $D = \frac{h^2}{2T}$.

III. Derivation

The rotational equations of motion (with external torques equal to zero) can be written in the following way:

$$J_1\dot{\omega}_1 - (J_2 - J_3)\omega_2\omega_3 = 0 \tag{1}$$
$$J_2\dot{\omega}_2 - (J_3 - J_1)\omega_3\omega_1 = 0 \tag{2}$$
$$J_3\dot{\omega}_3 - (J_1 - J_2)\omega_1\omega_2 = 0 \tag{3}$$

where, a dot over a variable expresses a derivative with respect to time.

Important for the discussion below will be the dynamical quantities T and h. The rotational kinetic energy is given by:

$$2T = J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2 \tag{4}$$

The magnitude of the angular momentum is given by:

$$h^2 = J_1^2\omega_1^2 + J_2^2\omega_2^2 + J_3^2\omega_3^2 \tag{5}$$

[1] If $J_1 = J_2 \neq J_3$ the solution derived here is not valid and one should refer to the case for an axi-symmetric body [3] pp. 53

The ratio of equations (5) and (4) plays an important role in the following derivation (it is denoted by D). The minimum and maximum values of D can be found by consequently multiplying equation (5) by J_1 and J_3 and then subtracting the resulting equations from (4).

$$J_2(J_1 - J_2)\omega_2^2 + J_3(J_1 - J_3)\omega_3^2 = 2T(J1 - D) \tag{6}$$
$$J_1(J_1 - J_3)\omega_1^2 + J_2(J_2 - J_3)\omega_2^2 = 2T(D - J3) \tag{7}$$

The left-hand side expressions are non-negative, hence $J_1 \geq D \geq J_3$. $D = J_1$ and $D = J_3$ correspond to *simple rotation* around the major and minor axis, respectively. The solutions for these cases are trivial, and will not be included here, hence it will be assumed that $J_1 > D > J_3$. Next, the above equations are solved for ω_3 and ω_1, respectively as a function of ω_2:

$$\omega_3^2 = \frac{J_2(J_1 - J_2)}{J_3(J_1 - J_3)}(b^2 - \omega_2^2) \tag{8}$$
$$\omega_1^2 = \frac{J_2(J_2 - J_3)}{J_1(J_1 - J_3)}(a^2 - \omega_2^2) \tag{9}$$

where, a and b are non-negative constants and are defined as:

$$a^2 = \frac{2T(D - J_3)}{J_2(J_2 - J_3)} \qquad b^2 = \frac{2T(J_1 - D)}{J_2(J_1 - J_2)}$$

Substitution of (8) and (9) in (2) leads to:

$$J_2\dot{\omega}_2 + (J_1 - J_3)s_1\sqrt{\frac{J_2(J1 - J_2)}{J_3(J_1 - J_3)}(b^2 - \omega_2^2)}s_3\sqrt{\frac{J_2(J_2 - J_3)}{J_1(J_1 - J_3)}(a^2 - \omega_2^2)} = 0 \tag{10}$$

where, s_1 and s_3 are constants that account for the sign of the two square roots (they can be determined from the signs of the initial conditions $\hat{\omega}_1$ and $\hat{\omega}_3$. $s_1 = \text{sign}(\hat{\omega}_1)$ and $s_3 = \text{sign}(\hat{\omega}_3)$ [2]). Rearranging the above equation leads to:

$$\frac{\frac{d\omega_2}{dt}}{\sqrt{(a^2 - \omega_2^2)(b^2 - \omega_2^2)}} = -s_1 s_3\sqrt{\frac{(J_1 - J_2)(J_2 - J_3)}{J_1 J_3}} \tag{11}$$

Equation (11) has to be integrated in order to obtain ω_2:

$$\int \frac{d\omega_2}{\sqrt{(a^2 - \omega_2^2)(b^2 - \omega_2^2)}} = \int -s_1 s_3\sqrt{\frac{(J_1 - J_2)(J_2 - J_3)}{J_1 J_3}}dt + C \tag{12}$$

where, C is an integration constant. Note that, the expression integrated on the right-hand side is constant, hence after integration one obtains:

$$\int \frac{d\omega_2}{\sqrt{(a^2 - \omega_2^2)(b^2 - \omega_2^2)}} = -s_1 s_3\sqrt{\frac{(J_1 - J_2)(J_2 - J_3)}{J_1 J_3}}t + C \tag{13}$$

[2] $\text{sign}(\hat{\omega}_i)$ denotes the sing of the initial condition $\hat{\omega}_i$

with t being time. Now (for convenience) we set $C = s_1 s_3 \sqrt{\frac{(J_1-J_2)(J_2-J_3)}{J_1 J_3}} t_0$, where t_0 is a new constant (to be evaluated shortly from the given initial conditions for ω). The above equation becomes:

$$\int \frac{d\omega_2}{\sqrt{(a^2-\omega_2^2)(b^2-\omega_2^2)}} = -s_1 s_3 \sqrt{\frac{(J_1-J_2)(J_2-J_3)}{J_1 J_3}}(t-t_0) \tag{14}$$

The integral on the left-hand side of (14) is an elliptic integral of the first kind (see Appendix 1), and in order to solve it, three cases have to be distinguished (the reason for this will become apparent later):

a) $a < b$ or $D < J_2$
b) $a > b$ or $D > J_2$
c) $a = b$ or $D = J_2$

Cases a) and b) correspond to polhodes which envelop the major and minor axis, respectively. c) is a limiting case, which separates the polhodes from cases a) and b) (for more information see [3]). Case a) will be considered first.

A. Solution for Case a)

Equation (14) can be transformed as follows:

$$\int \frac{\frac{d\omega_2}{a}}{\sqrt{(1-\frac{\omega_2^2}{a^2})(1-\frac{a^2}{b^2}\frac{\omega_2^2}{a^2})}} = -b s_1 s_3 \sqrt{\frac{(J_1-J_2)(J_2-J_3)}{J_1 J_3}}(t-t_0) \tag{15}$$

or

$$\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = -s_1 s_3 P(t-t_0) \tag{16}$$

with the following abbreviations (b is included in P):

$$x = \frac{\omega_2}{a} \quad k = \frac{a}{b} \quad P = \sqrt{\frac{2T(J_1-D)(J_2-J_3)}{J_1 J_2 J_3}} \tag{17}$$

The solution to the elliptic integral (of the first kind) in (16) has the form

$$x = -s_1 s_3 sn(P(t-t_0), k) \tag{18}$$

where, sn is a Jacobi elliptic function, with modulus k (it should satisfy $0 \le k \le 1$) and argument $P(t-t_0)$. The fact that the modulus k should be between 0 and 1 explains why three cases had to be distinguished [3]. Solving (18) for ω_2 gives:

$$\omega_2 = -a s_1 s_3 sn(P(t-t_0), k) \tag{19}$$

In (19) the constant t_0 is still undetermined. It can be calculated by setting $t = 0$ and using the initial condition for ω_2.

$$\hat{\omega}_2 = -a s_1 s_3 sn(-Pt_0, k) = a s_1 s_3 sn(Pt_0, k) \tag{20}$$

[3] In case a) since $a < b$ the restriction for k is satisfied. In case b) k will be defined as $\frac{b}{a}$. In case c) $k = 1$, and the solutions for cases a) and b) converge towards the same result.

In order to solve the above equation for t_0, we have to compute an elliptic integral (which is an inverse to the elliptic function sn, see Appendix 1) of the form:

$$Pt_0 = \int_0^{ul} \frac{d\omega_2}{\sqrt{(1-\omega_2^2)(1-k^2\omega_2^2)}} \tag{21}$$

where, the upper limit of the integral $ul = s_1 s_3 \frac{\hat{\omega}_2}{a}$.

The next step in the derivation is to determine ω_1 and ω_3. This is made by substituting equation (19) into (8) and (9) (The sign determining terms disappear, since ω_2^2 is needed).

$$\omega_1^2 = \frac{J_2(J_2-J_3)}{J_1(J_1-J_3)}[a^2 - \frac{2T(D-J_3)}{J_2(J_2-J_3)} sn(P(t-t_0),k)^2] \tag{22}$$

$$\omega_1^2 = \frac{J_2(J_2-J_3)}{J_1(J_1-J_3)} \frac{2T(D-J_3)}{J_2(J_2-J_3)}[1 - sn(P(t-t_0),k)^2] \tag{23}$$

$$\omega_1^2 = \frac{2T(D-J_3)}{J_1(J_1-J_3)} cn(P(t-t_0),k)^2 \tag{24}$$

and the solution for ω_3:

$$\omega_3^2 = \frac{J_2(J_1-J_2)}{J_3(J_1-J_3)}[b^2 - \frac{2T(D-J_3)}{J_2(J_2-J_3)} sn(P(t-t_0),k)^2] \tag{25}$$

$$\omega_3^2 = \frac{J_2(J_1-J_2)}{J_3(J_1-J_3)} \frac{2T(J_1-D)}{J_2(J_1-J_2)} - \frac{J_2(J_1-J_2)}{J_3(J_1-J_3)} \frac{2T(D-J_3)}{J_2(J_2-J_3)} sn(P(t-t_0),k)^2 \tag{26}$$

the above equation is multiplied by $\frac{J_3(J_1-J_3)}{2T(J_1-D)}$, where it was already assumed that $D \neq J_1$.

$$\omega_3^2 \frac{J_3(J_1-J_3)}{2T(J_1-D)} = 1 - k^2 sn(P(t-t_0),k)^2 \tag{27}$$

$$\omega_3^2 = \frac{2T(J_1-D)}{J_3(J_1-J_3)} dn(P(t-t_0),k)^2 \tag{28}$$

The solutions for ω_i (i = 1,2,3) are (see equations (19), (24), (28)):

$$\omega_1 = s_1 \sqrt{\frac{2T(D-J_3)}{J_1(J_1-J_3)}} cn(P(t-t_0),k) \tag{29}$$

$$\omega_2 = s_2 \sqrt{\frac{2T(D-J_3)}{J_2(J_2-J_3)}} sn(P(t-t_0),k) \tag{30}$$

$$\omega_3 = s_3\sqrt{\frac{2T(J_1 - D)}{J_3(J_1 - J_3)}}dn(P(t - t_0), k) \tag{31}$$

where, s_2 is a constant that determines the sign of the square root in (30). From (19) s_2 can be defined as $s_2 = -s_1 s_3$. The same relation can be found by substituting (29), (30) and (31) back in (2). Note that, $\dot{\omega}_2$ is given by:

$$\dot{\omega}_2 = s_2 P\sqrt{\frac{2T(D - J_3)}{J_2(J_2 - J_3)}}cn(P(t - t_0), k)dn(P(t - t_0), k)$$

The relation between s_1, s_2 and s_3 can be expressed as:

$$s_1 s_2 s_3 = -1 \tag{32}$$

Altogether four combinations of signs satisfying this relation are possible. This is in accordance with the fact that for each value of the parameter D two separate polhodes exist and each of them passes through zero in two different points (see Table I).

TABLE I
POSSIBLE SIGN COMBINATIONS

S_1	S_2	S_3
+	+	-
-	+	+
+	-	+
-	-	-

Which row form Table 1 has to be chosen can be determined uniquely from the signs of the initial conditions $\hat{\omega}_1$ and $\hat{\omega}_3$ (if the value of one of them is zero its sign is assumed to be "plus"). When s_1 and s_3 are known $s_2 = -s_1 s_3$.

The solution for case a) $(D < J_2)$ is complete. Following the same patern, the equations for case b) $(D > J_2)$ can be derived as well. Here, we give only the result:

B. Solution for Case b)

$$\omega_1 = s_1\sqrt{\frac{2T(D - J_3)}{J_1(J_1 - J_3)}}dn(P(t - t_0), k) \tag{33}$$

$$\omega_2 = s_2\sqrt{\frac{2T(J_1 - D)}{J_2(J_1 - J_2)}}sn(P(t - t_0), k) \tag{34}$$

$$\omega_3 = s_3\sqrt{\frac{2T(J_1 - D)}{J_3(J_1 - J_3)}}cn(P(t - t_0), k) \tag{35}$$

where, the modulus $k = \frac{b}{a}$, and the sign determining terms s_1, s_2 and s_3 are the same as in the previous case.

$$P = \sqrt{\frac{2T(J_1 - J_2)(D - J_3)}{J_1 J_2 J_3}}$$

C. Solution for Case c)

It is not necessary to integrate case c) since both solutions for cases a) and b) converge in the limit $D \rightarrow J_2$ towards the same result. Note that, here $k = 1$, and the elliptic functions become hyperbolic functions. Again s_1, s_2 and s_3 are the same as in the previous cases.

$$\omega_1 = s_1 \sqrt{\frac{2T(J_2 - J_3)}{J_1(J_1 - J_3)}} dn(P(t - t_0), 1) = s_1 \sqrt{\frac{2T(J_2 - J_3)}{J_1(J_1 - J_3)}} \frac{1}{\cosh(P(t - t_0))} \tag{36}$$

$$\omega_2 = s_2 \sqrt{\frac{2T}{J_2}} sn(P(t - t_0), 1) = s_2 \sqrt{\frac{2T}{J_2}} \tanh(P(t - t_0)) \tag{37}$$

$$\omega_3 = s_3 \sqrt{\frac{2T(J_1 - J_2)}{J_3(J_1 - J_3)}} cn(P(t - t_0), 1) = s_3 \sqrt{\frac{2T(J_1 - J_2)}{J_3(J_1 - J_3)}} \frac{1}{\cosh(P(t - t_0))} \tag{38}$$

IV. Angular velocity amplitude

As noted in [4], the maximum values for the amplitude of ω_1, ω_2 and ω_3 is equal to:

$$\omega_1^{max} = \sqrt{\frac{2T(D - J_3)}{J_1(J_1 - J_3)}} \qquad \omega_3^{max} = \sqrt{\frac{2T(J_1 - D)}{J_3(J_1 - J_3)}}$$

$$\begin{cases} \omega_2^{max} = \sqrt{\frac{2T(D-J_3)}{J_2(J_2-J_3)}} & (D < J_2) \\ \omega_2^{max} = \sqrt{\frac{2T(J_1-D)}{J_2(J_1-J_2)}} & (D > J_2) \\ \omega_2^{max} = \sqrt{\frac{2T}{J_2}} & (D = J_2) \end{cases}$$

these are precisely the terms before the elliptic functions in the solutions for ω_i (i = 1,2,3). The values for ω_1^{max}, ω_2^{max} and ω_3^{max} can be found alternatively by solving a maximization problem with constraints (see [4] pp. 106).

V. Appendix 1

This appendix contains a short introduction to elliptic functions and integrals. For more detailes see [7], [8].

A. Elliptic integrals

Elliptic integrals occur in many applications [8]. Many of them arose from attempts to solve mechanical problems. For example, the period of a simple pendulum was found to be related to an integral which expressed arc length but no form could be found in terms of "simple" functions. Another simple example is the integral that arises from the circumference of an ellipse. This example will be considered next.

Example:

The formula for computing arc length of any curve given by its parametric equations $x = f(t)$, $y = g(t)$, over the range $c \leq t \leq d$ is:

$$\int_c^d \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 dt} \tag{39}$$

The parametric equations of an ellipse are:

$$\begin{aligned} x &= a\sin(t) \\ y &= b\cos(t) \\ 0 <= t &<= 2\pi \\ \frac{dx}{dt} &= a\cos(t) \\ \frac{dy}{dt} &= -b\sin(t) \end{aligned}$$

where a and b are the ellipse radii on the major and minor axis, respectively. The eccentricity of an ellipse is defined as:

$$0 \leq e = \sqrt{1 - \frac{b^2}{a^2}} < 1 \tag{40}$$

Eccentricity 0 is in case of a circle. Substituting the above defined variables in (39) one obtains:

$$C = \int_0^{2\pi} \sqrt{a^2\cos(t)^2 + b^2\sin(t)^2 dt} \tag{41}$$

The symmetry of an ellipse can be used in order to rewrite the above equation as:

$$C = 4\int_0^{\pi/2} \sqrt{a^2\cos(t)^2 + b^2\sin(t)^2 dt} \tag{42}$$

Using the relation $\cos(t)^2 = 1 - \sin(t)^2$ one obtains:

$$C = 4\int_0^{\pi/2} \sqrt{1 - (1 - \frac{b^2}{a^2})\sin(t)^2 dt} \tag{43}$$

or

$$C = 4\int_0^{\pi/2} \sqrt{1 - e^2\sin(t)^2 dt} \tag{44}$$

The above integral is a complete elliptic integral of the second kind. It is called "complete", because the limits of integration are 0 and $\pi/2$. The eccentricity of the ellipse e is called the *modulus* of the integral in (44). By setting $v = \sin(t)$ ($dv = \cos(t)dt = \sqrt{1-t^2}dt$) one obtains [4]:

$$C = 4\int_0^{\sin(\pi/2)} \sqrt{\frac{1-e^2v^2}{1-v^2}}dt \tag{45}$$

Equation (44) is known as Legendre's form of complete elliptic integral of the second kind, (45) is known as Jacobi's form of this integral (they are equivalent).

The solution of the problem introduced in the above example, leads to an elliptic integral (for more examples see [8]). Three basic types of elliptic integrals exist. One of them was already defined. The other two are:

- Elliptic integral of the first kind:

$$u = F(\phi, k) = \int_0^{\phi} \sqrt{\frac{dt}{(1-k^2\sin(t)^2)}} \tag{46}$$

For generality, we consider ϕ to be the upper limit of the integral (if $\phi \neq \frac{\pi}{2}$ the integral is called *incomplete*). The *modulus* is denoted by k. Transforming (46) to Jacobi's form one obtains (compare with equation (16)):

$$u = F(\phi, k) = \int_0^{\sin(\phi)} \sqrt{\frac{dv}{(1-v^2)(1-k^2v^2)}} \tag{47}$$

- Elliptic integral of the third kind:

$$\int_0^{\phi} \frac{dt}{(1+n\sin(t)^2)\sqrt{(1-k^2\sin(t)^2)}} \tag{48}$$

$$\int_0^{\sin(\phi)} \frac{dv}{(1+nv^2)\sqrt{(1-v^2)(1-k^2v^2)}} \tag{49}$$

where, n is a constant known as the *elliptic characteristic.*

B. *Elliptic functions*

There are three basic elliptic functions. They arise from the inversion of the elliptic integral of the first kind and are denoted by: $sn(u,k)$, $cn(u,k)$ and $dn(u,k)$ (u is defined in (47)). The *modulus* k is assumed to be in the range $0 \leq k \leq 1$. It is obvious that:

$$sn(u,0) = \sin(u) \qquad cn(u,0) = \cos(u) \qquad dn(u,0) = 1$$

and

$$sn(u,1) = \tanh(u) \qquad cn(u,1) = \frac{1}{\cosh(u)} \qquad dn(u,1) = \frac{1}{\cosh(u)}$$

[4] Note that, when the integration variable is changed the upper limit of integration should be changed as well.

Hence, for $k = 0$ the elliptic functions become circular and for $k = 1$ they become hyperbolic.

The upper limit of integration in (46) (also called Jacobi amplitude) ϕ is defined in terms of $sn(u,k)$ as:

$$\sin(\phi) = sn(u,k) \tag{50}$$

The standard Jacobi elliptic functions satisfy the following identities:

$$sn(u,k)^2 + cn(u,k)^2 = 1$$
$$k^2 sn(u,k)^2 + dn(u,k)^2 = 1$$

Derivatives of the Jacobi elliptic functions are:

$$\frac{d\ sn(u,k)}{du} = cn(u,k)dn(u,k)$$
$$\frac{d\ cn(u,k)}{du} = -sn(u,k)dn(u,k)$$
$$\frac{d\ dn(u,k)}{du} = -k^2 sn(u,k)cn(u,k)$$

References

[1] http://www.aass.oru.se/Research/Learning/drdv_dir/reports/Euler_analyt.m
[2] T. Kane, P. Likins, and D. Levinson, "Spacecraft Dynamics," *McGraw-Hill*, 1983
[3] Wittenburg, "Dynamics of Systems of Rigid Bodies,", 1977
[4] P. Hughes, "Spacecraft Attitude Dynamics," *Wiley*, 1986
[5] J. Wertz, "Spacecraft Attitude Determination and Control," *Reidel*, 1978
[6] F. Rimrott, "Introductory Attitude Dynamics," *Springer-Verlag*, 1989
[7] A. Greenhill, "The applications of elliptic functions," 1892
[8] F. Bowman, "Introduction to elliptic functions with applications," 1961