Digital Image Processing

Part 3: Fourier Transform and Filtering in the Frequency Domain

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Course Book Chapter 4
2 Relation Between Spatial and Frequency Filters

- Derivatives and their Fourier Transform

\[
\mathcal{F} \left[ \frac{d^n f(x)}{dx^n} \right] = (ju)^n \mathcal{F}[f(x)]
\]

- Laplacian in the Fourier Domain

\[
\mathcal{F} [\nabla^2 f(x, y)] = -(u^2 + v^2) F(u, v)
\]

\[\Rightarrow H_{\text{Laplacian}} (u, v) = -(u^2 + v^2)\]
2 Relation Between Spatial and Frequency Filters

- Derivatives and their Fourier Transform

\[ \mathcal{F} \left[ \frac{d^n f(x)}{dx^n} \right] = (ju)^n \mathcal{F}[f(x)] \]

- Laplacian in the Fourier Domain

\[ \nabla^2 f(x, y) \Leftrightarrow -\left[ \left( u - \frac{M}{2} \right)^2 + \left( v - \frac{N}{2} \right)^2 \right] F(u, v) \]
2 Relation Between Spatial and Frequency Filters

- Laplacian in the Fourier Domain
2 Laplacian in the Fourier Domain

Laplacian in the Spatial Domain

\[
\mathcal{F}^{-1}[-(u^2 + v^2)(-1)^{u+v}(-1)^{x+y}]
\]
2 Deriving Spatial Filter Masks

General Idea

- select a filter in the frequency domain
- transform this filter to the spatial domain
- try to specify a small filter mask that captures the "essence" of the filter function
Other Filters

- **bandpass**
  - allows frequencies in a band between the two frequencies D0 and D1

- **bandstop**:
  - stops frequencies in a band between the two frequencies D0 and D1

- **non-symmetric filters**:
  - allow different frequencies in the u and v direction
Recovering Intrinsic Images, Homomorphic Filtering
3 Image Formation

- Reminder: Image Formation Model
  - illumination \( i(x,y) \) from a source
  - reflectivity \( r(x,y) = \text{reflection} / \text{absorption} \text{ in the scene} \)
  - \( f(x,y) = r(x,y) \cdot i(x,y) \)
3 Intrinsic Images

Intrinsic Images

"midlevel description" of scenes

- proposed by Barrow and Tenebaum
- not a full 3D description of the scene
- viewpoint dependent
- physical causes of changes in illumination are not made explicit
3 Intrinsic Images

- Intrinsic Images
  - "midlevel description" of scenes
    - proposed by Barrow and Tenebaum
  - "The observed image is a product of two images: an illumination image and a reflectance image."
3 Intrinsic Images

- "midlevel description" of scenes
- (input) image is decomposed into two images …

from "Deriving Intrinsic Images From Image Sequences", Yair Weiss, Proc. ICCV 2001
3 Intrinsic Images

Intrinsic Images

- "midlevel description" of scenes
- (input) image is decomposed into two images ...
  - a reflectance image

from "Deriving Intrinsic Images From Image Sequences", Yair Weiss, Proc. ICCV 2001
3 Intrinsic Images

- Intrinsic Images
  - "midlevel description" of scenes
  - (input) image is decomposed into two images …
    - a reflectance image and
    - an illumination image

\[ \text{Input} = r(x,y) \times i(x,y) \]
Intrinsic Images

- "midlevel description" of scenes
- "The observed image is a product of two images: an illumination image and a reflectance image."
  - segmentation on the intrinsic reflectance should be much simpler than on the original image
  - 3D information can be obtained from the illumination picture

\[ \text{Input} = r(x,y) \times i(x,y) \]
3 Intrinsic Images

Intrinsic Images

"midlevel description" of scenes

(input) image is decomposed into two images ...
  a reflectance image and
  an illumination image

but: decomposition is an ill-posed problem
  number of unknowns is twice as high as the number of equations
  (for example: set \( i(x,y) = 1 \rightarrow r(x,y) = f(x,y) \))

\[
f(x, y) = i(x, y) r(x, y)
\]
3 Homomorphic Filtering

Idea: Separate Illumination and Reflectance

\[ f(x, y) = i(x, y) r(x, y) \]

- not separable directly …

\[ \mathcal{F}[f(x, y)] \neq \mathcal{F}[i(x, y)] \mathcal{F}[r(x, y)] \]

- … but the logarithm is separable

\[ z(x, y) \equiv \ln[f(x, y)] \]

\[ \mathcal{F}[z(x, y)] = \mathcal{F}[\ln[i(x, y)]] + \mathcal{F}[\ln[r(x, y)]] \]

\[ = Z(u, v) = F_i(u, v) + F_r(u, v) \]
3 Homomorphic Filtering

- Frequency Domain Approximation to Homomorphic Filtering – Assumption
  - illumination component varies slowly
  - reflectance component tends to vary abruptly
  ⇒ use filter that affects low- and high-frequency components in a different way (decreases influence of illumination, increases influence of reflectance)

\[ f(x, y) = i(x, y) r(x, y) \]
\[ \gamma_L < 1 \]
\[ \gamma_H > 1 \]
### 3 Homomorphic Filtering

**Idea: Separate Illumination and Reflectance**

\[
g(x, y) = e^{s(x, y)} = e^{i'(x, z)} e^{r'(x, z)}
\]

\[f(x, y) \xrightarrow{\ln} \text{DFT} \xrightarrow{H(u, v)} (\text{DFT}^{-1}) \xrightarrow{\exp} g(x, y)\]

\[
Z(u, v) = F_i(u, v) + F_r(u, v) = \mathcal{F}[\ln[i(x, y)]] + \mathcal{F}[\ln[r(x, y)]]
\]

\[
S(u, v) = H(u, v)Z(u, v) = H(u, v)F_i(u, v) + H(u, v)F_r(u, v)
\]

\[
s(x, y) = \mathcal{F}^{-1}[S(u, v)] = \mathcal{F}^{-1}[H(u, v)F_i(u, v)] + \mathcal{F}^{-1}[H(u, v)F_r(u, v)] = i'(x, y) + r'(x, y)
\]
3 Homomorphic Filtering

Example

- consider non-uniform illumination

![Image of a person and a disturbance pattern]
3 Homomorphic Filtering

Example

- histogram equalization does not perform well
3 Homomorphic Filtering

Example

- homomorphic filtering

homomorphic filtering
3 Remarks

Homomorphic Filtering Assumption True?
["Deriving Intrinsic Images From Image Sequences", Yair Weiss, Proc. ICCV 2001]

- edges due to illumination often have as high a contrast as those due to reflectance changes
  - possible solution: deriving intrinsic images from image sequences
3 Remarks

- Recovering Intrinsic Images from a Single Image
  

- use colour information

Original Image
3 Remarks

- Recovering Intrinsic Images from a Single Image
  

  - use colour information
    
    - find chromaticity changes by classifying image derivatives by thresholding scalar product of normalized RGB vector neighbours

Original Image

Shape Image
3 Remarks

- Recovering Intrinsic Images from a Single Image
  

- use colour information
  
  find chromaticity changes by classifying image derivatives by thresholding scalar product of normalized RGB vector neighbours

![Original Image](image1.png) ![Shape Image](image2.png) ![Reflectance Image](image3.png)
3 Remarks

- Recovering Intrinsic Images from a Single Image
  

  - use colour information
  - learn appearance models of shading patterns
    - classify gray-scale image

(a) Original Image
3 Remarks

- Recovering Intrinsic Images from a Single Image
  

  - use colour information
  
  - learn appearance models of shading patterns
    
    - classify gray-scale image

(a) Original Image        (b) Shading Image
3 Remarks

- Recovering Intrinsic Images from a Single Image
  - use colour information
  - learn appearance models of shading patterns
    - classify gray-scale image

(a) Original Image
(b) Shading Image
(c) Reflectance Image
3 Remarks

- Recovering Intrinsic Images from a Single Image
  

  - use colour information
  
  - learn appearance models of shading patterns
    
    - classify gray-scale image
  
  - propagate evidence (MRF model with learned parameters)
3 Remarks

Recovering Intrinsic Images from a Single Image


- use colour information +
- learn appearance models of shading patterns +
- propagate evidence (MRF model with learned parameters)
3 Remarks

- Recovering Intrinsic Images from a Single Image
  

  - use colour information +
  - learn appearance models of shading patterns +
  - propagate evidence (MRF model with learned parameters)

(a) Original Image  
(b) Shading Image
3 Remarks

Recovering Intrinsic Images from a Single Image


- use colour information +
- learn appearance models of shading patterns +
- propagate evidence (MRF model with learned parameters)
3 Remarks

- Recovering Intrinsic Images from a Single Image
  

(a) Original Image
3 Remarks

Recovering Intrinsic Images from a Single Image


(a) Original Image  (b) Shading Image
3 Remarks

Recovering Intrinsic Images from a Single Image


(a) Original Image  (b) Shading Image  (c) Reflectance Image
Properties of the Fourier Transform
4 Properties of the Fourier Transform

Translation

\[ f(x - x_0, y - y_0) \iff F(u, v)e^{-j2\pi(\frac{u x_0}{M} + \frac{v y_0}{N})} \]

a shift in \( f(x,y) \) does not affect \( |F(u,v)| \)
4 Properties of the Fourier Transform

- **Translation**
  - A shift in \( f(x,y) \) does not affect the spectrum \( |F(u,v)| \)
4 Properties of the Fourier Transform

- Distributive Over Addition
  \[ \mathcal{F} [f_1(x, y) + f_2(x, y)] = \mathcal{F} [f_1(x, y)] + \mathcal{F} [f_2(x, y)] \]

- Not Distributive Over Multiplication
  \[ \mathcal{F} [f_1(x, y) f_2(x, y)] \neq \mathcal{F} [f_1(x, y)] \mathcal{F} [f_2(x, y)] \]

- Scaling
  \[ af(x, y) \Leftrightarrow aF(u, v) \quad f(ax, by) \Leftrightarrow \frac{1}{|ab|} F(u / a, v / b) \]

- Rotation
  \[ f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0) \]
4 Properties of the Fourier Transform

Rotation

Rotating \( f(x,y) \) rotates \( F(u,v) \) by the same angle.
4 Properties of the Fourier Transform

- **Periodicity**
  \[ F(u, v) = F(u + M, v) = F(u, v + N) = F(u + M, v + N) \]
  \[ f(x, y) = f(x + M, y) = f(x, y + N) = f(x + M, y + N) \]
  - the discrete Fourier transform is periodic
  - also the inverse of the discrete Fourier transform is periodic

- **Conjugate Symmetry**
  \[ F(u, v) = F^*(-u, -v) \]
  \[ \Rightarrow |F(u, v)| = |F^*(-u, -v)| \]
  - the spectrum is symmetric about the origin
4 Properties of the Fourier Transform

Separability

\[ F(u, v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)} = \]

\[ = \frac{1}{M} \sum_{x=0}^{M-1} e^{-j2\pi ux/M} \frac{1}{N} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi vy/N} = \]

\[ = \frac{1}{M} \sum_{x=0}^{M-1} e^{-j2\pi ux/M} F(x, v) = \mathcal{F}_u[\mathcal{F}_v[f(x, y)]] \]

- \( F(x, v) \) is the Fourier transform along one row
- \( F(u, v) \) can be obtained by two successive applications of the simple 1D Fourier transform instead of by one application of the more complex 2D Fourier transform
## Properties of the Fourier Transform

### Fourier Transform

<table>
<thead>
<tr>
<th>Function</th>
<th>Fourier transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta(x, y) )</td>
<td>1</td>
</tr>
<tr>
<td>( \frac{\partial f}{\partial x}(x, y) )</td>
<td>( u\mathcal{F}(f)(u, v) )</td>
</tr>
<tr>
<td>( 0.5\delta(x + a, y) + 0.5\delta(x - a, y) )</td>
<td>( \cos 2\pi au )</td>
</tr>
<tr>
<td>( e^{-\pi(x^2+y^2)} )</td>
<td>( e^{-\pi(u^2+v^2)} )</td>
</tr>
<tr>
<td>( box_1(x, y) )</td>
<td>( \frac{\sin u \sin v}{u ; v} )</td>
</tr>
<tr>
<td>( f(ax, by) )</td>
<td>( \frac{\mathcal{F}(f)\left(\frac{u}{a}, \frac{v}{b}\right)}{ab} )</td>
</tr>
<tr>
<td>( \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \delta(x - i, y - j) )</td>
<td>( \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \delta(u - i, v - j) )</td>
</tr>
<tr>
<td>( (f \ast g)(x, y) )</td>
<td>( \mathcal{F}(f)\mathcal{F}(g)(u, v) )</td>
</tr>
<tr>
<td>( f(x - a, y - b) )</td>
<td>( e^{-i2\pi(au+bv)}\mathcal{F}(f) )</td>
</tr>
</tbody>
</table>

from "Computer Vision – A Modern Approach", Forsyth and Ponce, Prentice Hall, 2002
4 Correlation

Definition of Correlation

\[ f(x, y) \circ h(x, y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f^*(m, n)h(x + m, y + n) \]

- compare with convolution

\[ f(x, y) \ast h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n)h(x - m, y - n) \]

- also the need for padding
4 Correlation

Template Matching

\[ f(x, y) \odot h(x, y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f^*(m,n) h(x+m, y+n) \]

- \( f(x,y) \) is the image
- \( h(x,y) \) is a template
- if \( h(x,y) \) matches somewhere in \( f(x,y) \) the correlation will be maximal there

\[ f(x, y) \odot h(x, y) \Leftrightarrow F^*(u,v)H(u,v) \]

\[ f^*(x, y) \odot h(x, y) \Leftrightarrow F(u,v)H(u,v) \]
4 Correlation

Template Matching – Example 1

\[ f_1(x,y) \text{ – padded} \quad h(x,y) \text{ – padded} \]
4 Correlation

Template Matching – Example 1

$f_1(x,y)$ – padded

$\mathcal{F}^{-1}[F_1^*(u,v)H(u,v)]$ – rescaled
4 Correlation

Template Matching – Example 1

\[ \mathcal{F}^{-1} [F_1^*(u,v)H(u,v)]^4 \] – rescaled^4

\[ f_1(x,y) \] – padded
4 Correlation

Definition of Correlation (from previous slide)

\[ f(x, y) \circ h(x, y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f^*(m, n)h(x + m, y + n) \]

Correlation Theorem

\[ f(x, y) \circ h(x, y) \Leftrightarrow F^*(u, v)H(u, v) \]

\[ f^*(x, y) \circ h(x, y) \Leftrightarrow F(u, v)H(u, v) \]
4 Correlation

Template Matching – Example 2

\[ f_2(x,y) \text{ – padded} \]

\[ h(x,y) \text{ – padded} \]
4 Correlation

- Template Matching – Example 2

\[ f_2(x,y) \text{ – padded} \quad \mathcal{F}^{-1}[F_2^*(u,v)H(u,v)]^4 \text{ – rescaled}^4 \]
4 Correlation

Template Matching – Example 3

\[ f_3(x,y) \text{ – padded} \quad h(x,y) \text{ – padded} \]
4 Correlation

Template Matching – Example 3

$f_3(x,y)$ – padded

$\mathcal{F}^{-1} [F_3^*(u,v)H(u,v)]^8$ – rescaled

Filters as Templates

- Filters respond most strongly to patterns that look like the filter.
- The kernel looks like the effect it is intended to detect.
- Filtering has an analogy to computing a dot product, which measures similarity to the filter kernel and results in a stronger response in brighter areas.
- Filtering can be seen as changing the basis.
  - Convolution can be seen as changing the base of an image.
    - Base: vectors \( \delta \)-functions \( \rightarrow \) base: shifted versions of the filter.
    - This process will typically lose information (coefficients on the new base can be redundant) but it might expose image structure in a useful way.
Nyquist-Shannon Sampling Theorem
5 Nyquist-Shannon Sampling Theorem

- Aliasing / Undersampling, Moiré Pattern

[Images showing a brick wall with and without Moiré Pattern, indicating decrease in resolution]
How to avoid Aliasing Problems?

- sampling
  - continuous function (irradiance in the camera) $\rightarrow$ discrete grid
  - number of samples relative to the function seems important
    - a signal sampled too slowly is misrepresented by the samples

from "Computer Vision – A Modern Approach", Forsyth and Ponce, Prentice Hall, 2002
5 Nyquist-Shannon Sampling Theorem

- Sampling a Signal in 1D

- Reconstruction of the Original Continuous Signal
  - which sample rate?
  - how to derive the continuous signal from the samples?
  - how to model the sampling process?
Sampling a Signal in 1D

How to Model the Sampling Process?

- continuous model of a sampled signal needed
- → sum of delta functions (2D: "bed-of-nails function")
  - sampling process = multiplication with a sampling function $f_{\text{III}}(x)$
Considering Band-Limited Signals

- signal band-width: band/range of non-zero frequencies
- a band-limited signal is constrained in terms of how fast it can change
5 Nyquist-Shannon Sampling Theorem

Fourier Transform of a Sampled Signal

\[ f(x) \cdot f_{III}(x) \]

- sampling = multiplication with the sampling function in the spatial domain
5 Nyquist-Shannon Sampling Theorem

Fourier Transform of a Sampled Signal

sampling = multiplication with the sampling function in the spatial domain

equals convolution in the frequency domain

\[ f(x) \cdot f_{\text{III}}(x) \rightarrow F(u) \ast F_{\text{III}}(u) \]

Fourier transform of a "Dirac comb" is again a Dirac comb

(\rightarrow\text{Poisson summation})
5 Nyquist-Shannon Sampling Theorem

Fourier Transform of a Sampled Signal

- sampling = multiplication with the sampling function in the spatial domain
- equals convolution in the frequency domain
5 Nyquist-Shannon Sampling Theorem

- Fourier Transform of a Sampled Signal

- non-overlapping support of the "shifted Fourier Transforms"

→ we can reconstruct the signal from the sampled versions
## 5 Nyquist-Shannon Sampling Theorem

### Reconstruction of the Signal

From "Computer Vision – A Modern Approach", Forsyth and Ponce, Prentice Hall, 2002
Reconstruction of the Signal

but if support regions do overlap?

- we can't reconstruct the signal
- Fourier transform in the regions that overlap can't be determined
Nyquist-Shannon Sampling Theorem

Fourier Transform of a "Dirac comb"

\[ f_{III}(x) \xrightarrow{\mathcal{F}} F_{III}(u) \]

reciprocal behaviour of \( \Delta x \) and \( \Delta u \)
Nyquist-Shannon Sampling Theorem

- Sampling Theorem
  - There should be no overlap between the repetitions of the FT of the signal.
  - The sampling interval should be at least the double of the highest frequency (1/w) present in the signal.

\[ 2w \leq \frac{1}{\Delta x} \Rightarrow \Delta x \leq \frac{1}{2w} \]